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Left Quasi-Artinian Rings and Modules

A thesis submitted in Partial Fulfillment of the Requirements of Award of the M.SC . Degree in pure mathematics

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Contents

INTRODUCTION AND SUMMARY

One of the important problems of Modules and Ring theory is the determination of the structure of Modules (Ring) satisfying the descending chain conditions on submodules (left ideals) , such Modules (Rings) are usually called Artinian Modules(Rings) after E.Artin (1893-1962) who first realized their importance. The successful study of such class of Modules and rings pointed out that a theory of more general Modules (rings) is possible. In this work we study those modules (rings), which satisfy certain chain condition of a descending type, leading to a new class of left Quasi-Artinian modules (rings), which possesses several of the main properties of left Artinian modules (rings). The condition required is the following:

Let R be a ring without (Possibly with) identity, and let M be a left R-module. We say that M is a *left quasi-Artinian* if for every descending chain $N_1 \supseteq N_2 \supseteq \ldots$ Of R-submodules of M, there exists $m \in Z^+$ *such that*

$R^m N_m \subseteq N_n \quad \forall n$.

If $_R R$ is left quasi-Artinian, we say that R is left quasi-Artinian ring.

Chapter one is devoted to fix a number of basic definitions, and list some

Well-Known results in ring theory which are used through this dissertation

Chapter two of this thesis consists of three sections. **In the first one** we give definition and examples of left Quasi-Artinian rings and module , which is generalization of left Artinian modules (rings) it is also a generalization of nilpotent rings , then we consider the problem of finding some condition which are equivalent to the definition (Theorem 2.1.5 & Corollary 2.1.6) . Next, we study the relation between left Artinian and left Quasi-Artinian modules, in particular we show that if RM is left Artinian, then M is left Quasi-Artinian (Theorem 2.1.7). Then , we show that the class of left Quasi-Artinian module is S-closed (Theorem2.1.9), Q-closed (Theorem 2.1.10) and E-closed (Theorem 2.1.12) while the class of left Quasi-Artinian rings is neither S-closed nor E-closed but it is I-closed (Theorem 2.1.14) and Q-closed (Theorem 2.1.10). Finally, we show that also a finite direct sum of left Quasi-Artinian rings is a left Quasi-Artinian ring (Theorem 2.1.16) .

In section two, we study the ideals structure of left Quasi-Artinian rings. First , we generalize Brauer's Theorem concerning Artinian rings and idempotent elements . In particular we prove if I is a non-nilpotent left ideal in a left Quasi-Artinian ring , then I contains a nonzero idempotent (Theorem 2.2.1), we then show that if R is a semi-prime left Quasi-Artinian ring and I is a nonzero left ideal of R , then I generated by a nonzero idempotent element (Theorem 2.2.6) also , we show that R is left Quasi-Artinian ring if and only if R is a direct sum of left Artinian ring with identity and nilpotent ring (Theorem 2.2.8).Also

we prove : If R is left Quasi-Artinian ring and I is a minimal ideal of R then *ann(I)* is a maximal ideal of R (Theorem 2.2.12) . Next we characterize the prime radical in left Quasi-Artinian ring (corollary 2.2.15). Finally we show that if R is left Quasi-Artinian ring, then there exists a finite number of distinct proper prime ideals of R.

In the last section of this chapter , we study the ideal and submodules structure by consider modules over left Quasi-Artinian ring , we show that if R is left Quasi-Artinian ring and M be a left R-module , then every finitely generated left R-module is left Quasi-Artinian (Theorem 2.3.2) . Also , we show that if R is left Quasi-Artinian ring and M be a left R-module then , (i) *Soc(M)* is an essential in *M,* and (ii) *Rad(M)* is small in M .(Theorem 2.3.3) . Finally , we give another characterization of left quasi-Artinian ring and module, Namely the following : If R is a ring , $N = N(R)$, be the nil radical of R then , R is a left Quasi-Artinian ring if and only if N is nilpotent and each of the

R/N , N/N^2 , N^2/N^3 , ... is left Quasi-Artinian R-modules (Theorem 2.3.5).

Finally we remark that tow papers [2, 3] based on this work have submitted for publication.

Chapter (I)

BASIC CONCEPTS

In this chapter we collect some well- known results which are needed.

1.1 Definitions and Examples

Definition 1.1.1

Let R be a ring , A and B are ideals (left or right or two-sided) of R ,

Then , the sum $A+B = \{ a+b \mid a \in A , b \in B \}$ and the product

$AB = \left\{ \sum_{\text{finite}} a_i b_i \setminus a_i \in A , b_i \in B \right\}$ is an ideal (left, right) in R .

It is clear that , if A and B are ideals , then $AB \subseteq A \cap B$

However, if A and B are left ideals , then $AB \subseteq B$.

Definition 1.1.2

(a) An element e of a ring R is called an *idempotent* if $e^2 = e$

Note that ,

1) If R is a ring with identity and is $e \in R$ is an idempotent , then (1- e) also an idempotent .

2) For any two idempotent e and f in R , we have

$$Re \oplus Rf = Re \oplus R(f - fe)$$

(b) An element a of a ring R is called *nilpotent* element if $a^n = 0$, for some $n \in \mathbf{Z}^+$.

(c) A non-zero element a of a ring R is called *zero divisor* if

$ab = ba = 0$, for some $0 \neq b \in R$

It is clear that , if R is a ring with identity then , every nilpotent element or idempotent element not equal 1 is a zero divisor .

Definition 1.1.3

(a) A (left , right , two sided) ideal I of a ring R is said to be a *nil ideal* if each element of I is nilpotent .

$I^n = (0)$ **(b)** A (left , right , two sided) ideal I of a ring R is said to be *nilpotent ideal* if there exists a positive integer *n* such that ,

Note that ,

(1) $I^n = (0)$ if and only if for each choice of n elements $a_1 , a_2 , \ldots , a_n \in I$, $a_1 a_2 \ldots a_n = 0$; in particular $a^n = 0$, for all a in I .

(2) If R is a commutative ring , then every nilpotent element generates a nilpotent ideal

Example 1.1.4

(a) Let $R = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a , b , c \in \mathbf{Z} \right\}$ .Then

$I = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbf{Z} \right\}$ is an ideal of R and $I^2 = (0)$

, therefore I is nilpotent .

(2) Every nilpotent ideal is a nil ideal , but the converse need not be true as the following example shows .

Let p be a prime number and $R = \oplus \sum Z/(p^i)$, $i = 1,2, \ldots$ be the direct sum of the rings $Z/(p^i)$, then R contains non-zero elements , such as , $\left(0 + (p) \ , \ p + (p^2) , \ 0 + (p^3) \ , \ \ldots \right)$. Let I be the set of all nilpotent elements . Then I is an ideal in R , since R is commutative . So I is a nil ideal , but I is not nilpotent . For if $I^n = 0$ for some $n \in \mathbf{Z}^+$, and $n > 1$ then , the element

$x = (0 + (p) , 0 + (p^2) , \ldots , 0 + (p^n) , P + (P^{n+1}) , 0 + (p^{n+2}) , \ldots)$ is nil-potent and so it belongs to I . But $x^n \neq 0$, a contradiction . So I is not nilpotent .

Definition 1.1.5 [15 , p 11]

The sum of all nil ideal of a ring R is called *the nil radical of* R and is denoted by *N(R)* or *W(R)* .

Note that , *N(R)* is a nil ideal of R , which contains all nil ideals of R .

Definition 1.1.6

A ring R is said to be *semi-prime* ring , if it has no non-zero nilpotent (right , left , two-sided) ideals .

Examples 1.1.7

Let $R = (\mathbf{Z}, +, .)$ *And* $S = (\mathbf{Z}[x], +, .)$, then R and S are semi- is prime rings , while $K = (\mathbf{Z}_8, \oplus, \otimes)$ not semi-prime ring .

Definition 1.1.8

(a) An R-module M is said to be *simple* if its submodules are (0) and M itself .

(b) An R-module M is said to be *irreducible* if M is a simple R-module and $RM = \{ \sum r_i m_i \setminus r_i \in R , m_i \in M \} \neq 0$

It is clear that , if $RM = 0$, then for each $r \in R$ *and* $m \in M$, $rm = 0$, so for unital R-module M , $RM \neq (0)$ if $M \neq (0)$

Note that , if $M = {}_R R$. Then we have the following definition :

Definition 1.1.9

A (left , right , two-sided) ideal I of a ring R is said to be *minimal ideal* if $I \neq (0)$ and there exists no (left , right , two sided) ideal J of R such that $(0) \subset J \subset I$.

Note that ,

(1) If R is a ring with identity , then a minimal left ideal I is an irreducible left R-module .

(2) It is clear that if M is an irreducible unital R-module , then for all non-zero element x in M , $Rx = M$.

Theorem 1.1.10 [15]

If R semi-prime ring and I is a minimal left ideal of R . Then $I=Re$ for some idempotent e in R .

Definition 1.1.11

A ring R is said to be *simple ring* if $R^2 \neq (0)$ and R has no ideals other than (0) and R itself .

Note that , every commutative simple ring is a field .

Example 1.1.12

(a) Let $R = (\mathbf{Z}, +, .)$, then there are no minimal ideals in R , therefore there is no irreducible R-submodules in R .

(b) Let $R = (\mathbf{Z}_4, \oplus, \otimes)$ and $I = \langle 2 \rangle \lhd R$, then I is a minimal ideal in R .

(c) Let $R = (\mathbf{Z}_5, \oplus_5, \otimes_5)$, then R is a simple ring .

Definition 1.1.13

A (left , right , two-sided) ideal I in the ring R is said to be *maximal ideal* if $I \neq R$ and there exists no left (right , two-sided) ideal J of R such that $I \subset J \subset R$.

Note that ,

An ideal $I \neq R$ in a ring R is maximal if and only if R/I is simple ring . Hence if R is a commutative ring with identity , then

I is maximal ideal if and only if R/I is a field .

Definition 1.1.14

(a) An ideal P of a ring R is said to be *prime ideal* , if $AB \subseteq P$ where A and B are ideals in R then , $A \subseteq P \vee B \subseteq P$.

(b) A ring R is said to be *a prime ring* if the zero ideal is a prim ideal in R . Hence if $AB = (0)$, A and B are ideals in R then , $A = (0)$ or $B = (0)$

Note that ,

(i) If R is a commutative ring , then P is a prime ideal in R if and only if for all $a , b \in R$, $ab \in P$ implies that either $a \in P$ *or* $b \in P$.

(ii) An ideal P is a prime ideal of R if and only if R/P is a prime ring .

Examples 1.1.15

(a) Let $R = (\mathbf{Z}, +, .)$, and $I = <p>$, then I is a prime ideal of R and it is also a maximal ideal of R .

(b) Let $R = (\mathbf{Z}[x], +, .)$, and $I = \langle x \rangle$, then I is a prime ideal of R which is not maximal .

Definition 1.1.16

Let I be a non-empty subset of a ring R , then

$l(I) = \{ x \in R \mid xI = 0 \}$ is the left annihilator of I in R .

It is clear that

$l(I)$ is a left ideal of R . If $I = \{x\}$, we write $l(I) = l(x)$. We say that the left ideal A of R is a left annihilator if $A = l(I)$, for some subset I of R , The right $r(I)$ annihilator can be defined similarlly.

It well-Know that ,

1) $I \subseteq l(r(I))$, $I \subseteq r(l(I))$.

2) If $I \subseteq J$, then $l(I) \supseteq l(J)$

3) $A = l(I)$ if and only if $A = l(r(A))$

4) If I is a left ideal of R then , $l(I)$ is an ideal of R .

5) If I and J are any subset of R then ,

(i) $l(I) + l(J) \subseteq l(I \cap J)$,
(ii) $l(I \cup J) = l(J) \cap l(J)$.

Note that ,

If R is a commutative ring we write $ann(I)$ instead of $l(I)$ or $r(I)$ and we usually called it by the annihilator of I in R .

Remark 1.1.17

If R is a semi-prime ring and I is an ideal of R , then

(a) $r(I) = l(I)$

(b) If I is a left annihilator ideal in R , then

$I = l(r(I)) = r(l(I))$. So that I is also a right annihilator ideal in R and we call I an annihilator ideal .

(c) If M is a left R-module , and I is an ideal of R contained in the

annihilator of M. Then the lattices of R-submodules and R/I - submodules coincide .

Example 1.1.18

(a) Let $R = (\mathbf{Z}_6, \oplus, \otimes)$, $I = \langle 2 \rangle$ is an ideal of R, then $ann(I) = \langle 3 \rangle$.

(b) Let $R = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} \end{bmatrix} = (M_2(\mathbf{R}), +, .)$, and

$I = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbf{R} \right\}$ is an ideal of R, then

$$l_R(I) = \begin{bmatrix} 0 & \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} = \left\{ \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbf{R} \right\} .$$

Definition 1.1.19

Let M be an R-module . A submodule K of M is said to be *essential (or . large or dense)* R-submodule in M if for each nonzero submodule L of M we have $K \cap L \neq 0$

If $M = {}_R R$ Then , we said I is an essential (left ,right,two sided) ideal in R and is denoted by (I *ess* R)

Example 1.1.20

(a) Let $R = (\mathbf{Z}, +, .)$ .Then every nonzero ideal of R is essential .

(b) Let $R = (\mathbf{Z}_{12}, \oplus, \otimes)$ then , $\langle 2 \rangle$ *ess* R, but $\langle 3 \rangle$ is not essential since $\langle 3 \rangle \cap \langle 4 \rangle = 0$ and $\langle 4 \rangle \lhd R$.

Definition 1.1.21

Let M be an R-module . A submodule K of M is said to be *small* R-submodule in M if for each nonzero submodule L of M

$$K + L = M \quad \text{implies} \quad L = M$$

If $M = R$ then , we said K is a *small* (left , right , two-sided) ideal in R .

Example 1.1.22

(a) Let $R = (\mathbf{Z}_{12}, \oplus, \otimes)$. Then $\langle 6 \rangle$ is a small ideal in R, but $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 4 \rangle$ are not small ideals in R since $\langle 2 \rangle + \langle 3 \rangle = R = \langle 4 \rangle + \langle 3 \rangle$

(b) Let $R = \begin{bmatrix} \mathbf{Z_2} & \mathbf{Z_2} \\ 0 & \mathbf{Z_2} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbf{Z_2} \right\}, +, . \right)$

Then $I = \begin{bmatrix} 0 & \mathbf{Z_2} \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mid x \in \mathbf{Z_2} \right\}$ is a *small* ideal in R, but

$J = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Z_2} \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbf{Z_2} \right\}$ is not *small* right ideal in R, since

$J + N = R$, where $N = \begin{bmatrix} Z_2 & Z_2 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} a & \mathrm{b} \\ 0 & 0 \end{bmatrix} \mid a, \mathrm{b} \in Z_2 \right\}$ is a right ideal

in R .

Definition 1.1.23

Let R be a ring and M is a left R-module . Then ,the *socle* of M is

(a) $Soc_l(M) = \sum \{ K \leq M \mid K \text{ is simple lef } R\text{-submodule in } M \}$

(b) $Soc_r(M) = \sum \{ K \leq M \mid K \text{ is simple right } R\text{-submodule in } M \}$

$$(c)\ Soc\,(M) = \sum\{K \le M \mid K \text{ is minimal (simple) in } M\}$$
$$= \bigcap\{L \le M \mid L \text{ is essential in } M\}$$

Note that**,** [4 , p119]

Soc(M) of M is the largest submodule of M that is contained in every essential submodule of M . In general , *Soc(M)* need not be essential in M

For example:

$M = {}_{\mathbf{Z}}\mathbf{Z}$ then , $Soc({}_{\mathbf{Z}}\mathbf{Z}) = 0$, which is not essential in $\mathbf{Z}$

Theorem 1.1.24 [4 , p 121]

Let M be a left R-module , then

$$Soc(M) = M \quad \text{if and only if} \quad M \text{ is semi-simple .}$$

Remark 1.1.25

(a) $Soc_l\,(M) \ne Soc_r\,(M)$ As the following example shows,

$$\text{Let } R = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \setminus a, b, c \in F\ ,\ F \textit{ is a field} \right\}, +\ ,\ . \right), \text{ then}$$

$$Soc_r\,({}_RR) = \left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \mid x, y \in F \right\} \quad \text{while,}$$
$$Soc_l\,({}_RR) = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in F \right\}$$

However , if R be a semi-prime ring then , $Soc_r\,(R) = Soc_l\,(R)$

(b) If M be left R - module and $K \le M$, then $Soc(K) = K \cap Soc(M)$

(c) *Soc(M) ess M* if and only if $Soc(K) \ne 0$ for evry nonzero submodule K of M .

(d) If there are no minimal submodules in M we put $Soc(M)=0$

Dual to the socle we define :

Definition 1.1.26

If M is a left R-module . Then ,the radical of M is defined as

$$\begin{aligned} Rad\,(M) &= \bigcap \{ K \mid K \text{ is maximal submodule in } M \} \\ &= \sum \{ L \mid L \text{ is small submodule in } M \} \end{aligned}$$

Note that , If M has no maximal submodules we put $Rad(M)=M$. In general $Rad(M)$ *need* not be small in M .

Theorem 1.1.27 [5]

Let N be a small submodule of an R-module M . Then M is finitely generated if and only if M/N is finitely generated .

Example 1.1.28

(a) Let $M = {}_{\mathbf{Z}}\mathbf{Q}$ then , $Rad({}_{\mathbf{Z}}\mathbf{Q}) = \mathbf{Q}$ and $Soc({}_{\mathbf{Z}}\mathbf{Q}) = 0$, since M has no maximal and no minimal Z-submodules .

(b) Let $M = {}_{\mathbf{Z}}\mathbf{Z}$ then , $Rad({}_{\mathbf{Z}}\mathbf{Z}) = Soc({}_{\mathbf{Z}}\mathbf{Z}) = 0$, since M has no small and no minimal submodules . On the other hand ,

$Rad({}_{\mathbf{Q}}\mathbf{Q}) = 0$ and $Soc({}_{\mathbf{Q}}\mathbf{Q}) = \mathbf{Q}$.

Definition 1.1.29

The intersection of all prime ideals of a ring R is called the *prime radical of* R and it is denoted by $rad\,(R)$.

Note that ,

(a) $rad\,(R) = (0)$ if and only if R is semi-prime ring

(b) If R is a commutative ring then ,

$rad(I) = \{ r \mid r^n \in I \ \ for\ some\ \ n \in \mathbf{N} \}$ and is usually denoted by $\sqrt{I}$

Example 1.1.30

(a) Let $R = (\mathbf{Z}, +, .)$, $I = \langle 12 \rangle$, then $rad(R) = (0)$ and $rad(I) = \langle 6 \rangle$

(b) Let $R = (\mathbf{Z}[x], +, .)$, then $rad\,(R) = (0)$.

(c) *Let* $R = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a , b , c \in \mathbf{Z} \right\} ,+ , . \right)$, then

$$rad\,(R) = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & 0 \end{bmatrix} = \left(\left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \mid x \in \mathbf{Z} \right\} ,+ , . \right).$$

Definition 1.1.31

(a) If M is a left R-module then , $A\,(M) = \{ x \in R \setminus xM = 0 \}$ is *an annihilator of* M *in* R .

It is clear that *,* $A(M)$ is an ideal of R

(b) Let R be a ring then , the *Jacobson radical* of R is the set :

$$J(R) = \cap \{ A(T) \mid T \text{ is an (simple) irreducible submodule} \}$$

If R has no irreducible submodules we put $J(R)=R$. Since $A(M)$ is two-sided ideal of R , it follows that $J(R)$ is an ideal of R .

If $M={}_R R$, then $Rad\,{}_R R = J(R)$.

Note that ,

(a) If R is a ring with identity then ,

$$J(R) = \cap \{ M \setminus M \text{ is maximal left ideal of } R \}.$$

(b) $rad(R) \subseteq N(R) \subseteq J(R)$.

(c) If $a \in R$ such that $RaR \subseteq J(R)$ then , $a \in J(R)$.

<u>Definition 1.1.32</u> [6 , p 157]

The ring R is said to be *semi-simple ring* if $J(R) = (0)$.

<u>Example 1.1.33</u>

(a) Let $R = (\mathbf{Z}[x], +, .)$, then $I = < p , x >$ is a maximal ideal of R for every prime number p . Therefore , $J(R) = \cap < p , x > = < x >$.

(b) Let $R = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a , b , c \in \mathbf{R} \right\} , + , . \right)$, then

$$J(R) = \left(\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a , b \in \mathbf{R} \right\} , + , . \right) \cap \left(\left\{ \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix} \mid b , c \in \mathbf{R} \right\} , + , . \right)$$

$$= \left(\left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbf{R} \right\} , + , . \right)$$

(c) Let $R = (\mathbf{Z}_{30} , \oplus , \otimes)$, then $J(R) = \langle 2 \rangle \cap \langle 3 \rangle \cap \langle 5 \rangle = 0$. Hence R is semi-simple .

Remark 1.1.34 [4, p171]

For a ring R with identity the following statements are equivalent :

(1) $R/J(R)$ is semisimple .

(2) For every left R-module M ,. $Soc(M) = r_M(J(R))$

1.2 Artinian Modules and Rings :

Definition 1.2.1

An R-module M is called ***Artinian*** if its submodules satisfy the descending chain condition (d.c.c) i.e , every descending chain $N_1 \supseteq N_2 \supseteq \dots \supseteq N_n \supseteq \dots$ of submodules of M becomes stationary after finitely many steps , that is there exists $m \in \mathbf{Z}^+$, such that , $N_m = N_k$, for all $k \geq m$.

A ring R is called ***left (right) Artinian*** if R regarded as left (right) R-module is an Artinian .If R is a commutative then , the concept of left Artinian and right Artinian are coincide .

Example 1.2.2

(a) Any simple ring R is Artinian .

(b) Every finite ring is Artinian .

(c) Let $R = (\mathbf{Z}, +, .)$ then , R is not Artinian since for all $n \in \mathbf{Z}^+$, we have an infinite descending chain $\langle 2 \rangle \supseteq \langle 2^2 \rangle \supseteq \ldots \supset \langle 2^n \rangle \supseteq \ldots$ of ideals of R .

(d) Let $R = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a \in \mathbf{Q} , b , c \in \mathbf{R} \right\} ,+ , . \right)$, then

R is not left Artinian because if

$$\mathbf{Q}(t_1,t_2,\ldots) = \left\{ a + \sum_{i=1} b_i t_i \mid a,b_i \in \mathbf{Q} ,t_i \in R \right\} \quad \text{, then}$$

$\mathbf{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{p}, \ldots) \leq \mathbf{R}$, and

$\begin{bmatrix} 0 & \mathbf{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix}$ is a left ideal of R . Also ,

$\begin{bmatrix} 0 & \mathbf{Q}(\sqrt{3}, \sqrt{5}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix}$ is a left ideal of R , and as left ideals we have , $\begin{bmatrix} 0 & \mathbf{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix} \supseteq \begin{bmatrix} 0 & \mathbf{Q}(\sqrt{3}, \sqrt{5}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix}$

Continuing this process , we get an infinite descending chain of left ideals

$\begin{bmatrix} 0 & \mathbf{Q}(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix} \supseteq \begin{bmatrix} 0 & \mathbf{Q}(\sqrt{3}, \sqrt{5}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix} \supseteq$

$\begin{bmatrix} 0 & \mathbf{Q}(\sqrt{5}, \sqrt{7}, \ldots, \sqrt{p}, \ldots) \\ 0 & 0 \end{bmatrix} \supseteq \ldots$ which gives that the matrix ring R is not left Artinian .

Definition 1.2.3

A class of rings $\mathcal{X}$ is said to be :

(a) S - closed , if $R \in \mathcal{X}$, and $S \leq R$, then $S \in \mathcal{X}$

(b) I - closed , if $R \in \mathcal{X}$ and $J \lhd R$, then $J \in \mathcal{X}$.

(c) Q - closed , if $R \in \mathcal{X}$ *and* $I \lhd R$, *then* $R/I \in \mathcal{X}$.

(d) E - closed , if I and $R/I \in \mathcal{X}$, then $R \in \mathcal{X}$

Example 1.2.4

Let $\mathcal{X}$ be the class of all left Artinian rings , then

(a) If $R \in \mathcal{X}$ and $I \lhd R$ then , $I \notin \mathcal{X}$ as the following example show :

Let $R = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ 0 & \mathbf{Q} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, \mathrm{b} \in \mathbf{R} \ , c \in \mathbf{Q} \right\} ,+ , . \right)$ and

$I = \begin{bmatrix} 0 & \mathbf{R} \\ 0 & 0 \end{bmatrix} = \left(\left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbf{R} \right\} ,+ , . \right)$ then , I is an ideal of R

and it is well known that , $R \in \mathcal{X}$, but $I \notin \mathcal{X}$.

(b) If $R \in \mathcal{X}$ and $S \leq R$ then , S need not be in $\mathcal{X}$ for example ,

$\mathbf{Q} \in \mathcal{X}$ and $\mathbf{Z} \leq \mathbf{Q}$ but $\mathbf{Z} \notin \mathcal{X}$.

***Finally* , we collect the following well-known results which we shall need .**

Theorem 1.2.5

Let R is a left Artinian ring , then

(a) Any nil (right , left , two-sided) ideal of R is nilpotent .

(b) $J(R) = rad(R) = N(R)$.

(c) If R is a commutative ring with identity then , every proper prime ideal of R is maximal **.**

Theorem 1.2.6 ***(Brauer's Theorem)*** [6 , p262]

If R is left Artinian ring , then every nonzero non-nilpotent left ideal of R contains a non-zero idempotent element .

Theorem 1.2.7 ***(Weederburn-Artin Theorem)*** [6 , p266]

If R be a semi-simple left (right) Artinian ring then , R is the finite direct sum of its minimal ideals each of which is a simple left Artinian ring.

Theorem 1.2.8 [6 , p 266]

If R is a semi-simple left Artinian ring viewed as rings , then

(a) Each ideal of R is itself a semi-prime left Artinian ring .

(b) Any minimal ideal of R is a simple ring .

(c) R has an identity element .

(d) Any nonzero left ideal I of R is generated by an idempotent element that is , $I = Re$ for some idempotent e in R .

Note that *,* If R is semi-prime Artinian , then R is semi-simple Artinian .

Theorem 1.2.9 [13]

Let $B \subseteq A$ and $C \subseteq A$ be unitary subring of a ring A , and note that we can view A as a right B-module , or as a left C-module . Let

$$R = \left\{ \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \mid a \in A , b \in B , c \in C \right\} \subseteq M_2(A)$$ and let

$$I = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \mid a \in A \right\} \subseteq R$$, then I is Artinian as left R-module

if and only if ${}_C A$ is Artinian.

Chapter (II)

Left Quasi-Artinian Rings and Modules

This chapter consists of three sections, in the first we give definitions ,examples and basic properties of the left Quasi-Artinian rings and modules while in the last two sections we study the internal structures of ideals and submodules and we give some classification of such class of rings and modules .

2.1 : Definitions and Basic properties

In this part we define the *left Quasi-Artinian rings and modules* , which is a generalization of Artinian rings and modules , it is also generalization of nilpotent rings , and then we consider the problem of finding conditions which are equivalent to the definition. Then , we prove that if *RM* is a left Artinian ,then M is left Quasi-Artinian, then we show that the class of left Quasi-Artinian modules is S-closed , Q-closed and E-closed while the class of left Quasi-Artinian rings is neither S-closed nor E-closed , but it is I-closed and Q-closed . Finally , we show also that a finite direct sum of left Quasi-Artinian rings is a left Quasi-Artinian .

Definition 2.1.1

Let R be a ring and let M be a left R-module.

We say that M is a *left quasi-Artinian* if for every descending

chain $N_1 \supseteq N_2 \supseteq \ldots$ of R-submodules there exists $m \in \mathbf{Z}^+$ *such that*

$R^m N_m \subseteq N_n \quad \forall n$.

If ${}_R R$ is left quasi-Artinian , we say that R is left quasi-Artinian ring.

Examples 2.1.2

(a) Left Artinian rings or modules are left Quasi-Artinian.

(b) Nilpotent rings are left Quasi-Artinian . Hence

$R = \begin{bmatrix} 0 & \mathbf{Q} \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$ are left Quasi-Artinian rings , since

$R^2 = 0$ and $S^2 = 0$, but neither R nor S is left Artinian since , for

each $k \in \mathbf{Z}^+$, $I_k = \left\{ \begin{bmatrix} 0 & m2^k \\ 0 & 0 \end{bmatrix} \mid m \in \mathbf{Z} \right\}$ is a left ideal of R and

$I_k \supsetneqq I_{k+1}$ Thuse , there exists an infinite properly descending chain of left

ideals of R , namely, $I_1 \supsetneqq I_2 \supsetneqq \ldots$. Similarly, S is not Artinian , since

$J_k = \left\{ \begin{bmatrix} 0 & 0 \\ m2^k & 0 \end{bmatrix} \mid m \in \mathbf{Z} \right\}$ form an infinite descending chain of left.

ideals of S .

(c) If M is a left R-module and R is a nilpotent ring , then M is left Quasi-Artinian R-module .

(d) Let $R = \begin{bmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & 0 \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a , b \in \mathbf{Q} \right\} , + , . \right)$, then

R is a non-nilpotent ring which is left Quasi-Artinian ring , since for each R are

$k \in \mathbf{Z}^+$ the left ideals of of the form

$$\begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix} , J_K = \left(\left\{ \begin{bmatrix} 0 & 0 \\ \dfrac{r}{p^K} & 0 \end{bmatrix} \mid r \in \mathbf{Z} \right\}, + ,. \right) \text{ and}$$

$$I_k = \left(\left\{ \begin{bmatrix} 0 & 0 \\ r\,p^k & 0 \end{bmatrix} \mid r \in Z \right\} , + , . \right) \text{, for any prime number } p,$$

therefore the descending chains left ideals in R are $I_1 \supsetneqq I_2 \supsetneqq \cdots$

Then , take $m=1$ we have $RI_1 = 0 \subseteq I_n$ for all n .Hence $R^m I_m \subseteq I_n$

for all n and R is left Quasi-Artinian , but it is not a left

Artinian ring , for in particular $I_k = \left\{ \begin{bmatrix} 0 & 0 \\ r2^k & 0 \end{bmatrix} \mid r \in \mathbf{Z} \right\}$ then , I_K

is a left ideal of R for each $k \in \mathbf{Z}^+$ and $I_1 \supsetneqq I_2 \supsetneqq \cdots$ is an infinite

descending chain of left ideals of R .

***Next* , we prove the following :**

<u>Lemma 2.1.3</u>

(a) Let R be a ring and M is a left R-module . If $RM = 0$ then , M is a left Quasi-Artinian.

(b) Any left Quasi-Artinian ring with identity is left Artinian .

<u>proof :</u>

(a) Since for every descending chain $N_1 \supseteq N_2 \supseteq \cdots$ of R-sub-

modules of M there exists an $m = 1 \in \mathbf{Z}^+$ such that , $RN_1 \subseteq RM = 0$.

Therefore , $RN_1 \subset N_n$, for all n . Hence M is left Quasi-Artinian

(b) Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of left ideals of R ,

since , R is left Quasi-Artinian then , there exists $m \in \mathbf{Z}^+$ such that ,

$R^m I_m \subseteq \cap I_n \subseteq I_n$, for all n .But R has an identity element , hence

$R^m I_m = I_m \subseteq I_n$ for all n , but $I_n \subseteq I_m$ $\forall\, n \geq m$. Therefore $I_m = I_n$,

$\forall\ n \geq m$ and R is left Artinian .

□

Example 2.1.4

(a) $R = (\mathbf{Z}, +, .)$ is not left Quasi-Artinian .

(b) The statement of the Hilbert Basis Theorem [6] is no longer true in left Quasi-Artinian ring , for example :

If R is left Quasi-Artinian ring then , the polynomial ring $R[x]$ need not be left Quasi-Artinian ring for example if $R = \boldsymbol{F}$ ($\boldsymbol{F}$ is any field) such that *char* $\boldsymbol{F}$ = 0 then , $\boldsymbol{F}$ is left Quasi-Artinian (since F is Artinian) but , $\boldsymbol{F}[x]$ over any field $\boldsymbol{F}$ is not left Quasi-Artinian ring , by Lemma 2.1.3 since $\boldsymbol{F}[x]$ has an identity element .

(c) If $M = \boldsymbol{F}[x]$ is any $\boldsymbol{F}$-module , then M is not left Quasi-Artinian $\boldsymbol{F}$-module since M is not left Artinian $\boldsymbol{F}$-module .

(d) $\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle|\ a \in \mathbf{Q} \ , b\, ,\ c \in \mathbf{R} \right\} , + , . \right)$ is a ring

with identity which is not left Quasi-Artinian , since it is not left Artinian see example 1.2.2(d)

(e) Let $R = \begin{bmatrix} \mathbf{Z} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$, then M is a left R-module .By Theorem 1.2.9 , M is an Artinian left R-module if and only if ${}_{\mathbf{Q}}\mathbf{Q}$ is Artinian . But ${}_{\mathbf{Q}}\mathbf{Q}$ is Artinian , hence M is Artinian left R-module and therefore M is left Quasi-Artinian R-module .

(f) Let $R = \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$ and $M = \begin{bmatrix} 0 & 0 \\ \mathbf{R} & 0 \end{bmatrix}$, then M is a left R-module by Theorem1.2.9 M is Artinian left R-module if and only if ${}_{\mathbf{Q}}\mathbf{R}$ is Artinian but ${}_{\mathbf{Q}}\mathbf{R}$ is not Artinian as $\mathbf{Q}\left(\sqrt{p}\,,\sqrt[3]{p}\,,\sqrt[4]{p}\,,\,\ldots\right) \supseteq \mathbf{Q}\left(\sqrt[3]{p}\,,\sqrt[4]{p}\,,\ldots\right) \supseteq \ldots$. is an infinite descending chain of left R-submodules of M , therefore M is not Artinian as left R-module and since R has an identity, then M is not left Quasi-Artinian R-module .

(g) Let $M = (\mathbf{Q}, +, .)$ be a **Z**-module . Then **Q** is not Quasi-Artinian **Z**-module since **Q** is not Artinian **Z**-module .

Now **, we consider the problem of finding conditions which are equivalent to a Definition 2.1.1 . In particular we prove the following :**

Theorem 2.1.5

Let M be a left R-module .Then the following conditions are equivalent :

(a) In every non-empty collection ς of left R-submodules of M,

such that if $k \in \varsigma$ implies $Rk \in \varsigma$ there exists a minimal element .

(b) For every descending chain of left R-submodules $N_1 \supseteq N_2 \supseteq \ldots$

there exists $m \in \mathbf{Z}^+$ such that a descending chain $R^m N_1 \supseteq R^m N_2 \supseteq \ldots$

terminates .

(c) M is left Quasi-Artinian .

(d) For every non-empty collection ς of left R-submodules of M,

there exists $N \in \varsigma$ and $m \in \mathbf{Z}^+$ such that , $R^m N \subseteq K$, for any

$K \in \varsigma$, $K \subseteq N$.

Proof :

(a)$\Rightarrow$ (b) Suppose that $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ is a descending chain of left R-submodules of M , but the descending chain $R^m N_1 \supseteq R^m N_2 \supseteq \ldots \supseteq R^m N_n \supseteq \ldots$ of left R-submodules of M does not terminate for all $m \in \mathbf{Z}^+$. Therefore , the collection $\varsigma = \{N_1, N_2, \ldots, RN_1, RN_2, \ldots, R^m N_1, R^m N_2, \ldots\}$ is nonempty collection of R-submodules and for $N \in \varsigma$, we have $RN \in \varsigma$. Therefore , it has minimal element , which is a contradiction .

(b)$\Rightarrow$(c) Let $N_1 \supseteq N_2 \supseteq ... \supseteq N_n \supseteq ...$ be any descending chain of left R-submodules of M then there exists $m \in \mathbf{Z}^+$ such that $R^m N_1 \supseteq R^m N_2 \supseteq ... \supseteq R^m N_n \supseteq ...$ form a descending chain of left R-submodules of M and by (b) there exists $s \in \mathbf{Z}^+$ such that $R^m N_s = R^m N_n$ for all $n \geq s$, but $R^m N_s \subseteq N_n$ for all $n \geq s$. Take t = max $\{m, s\}$ then $R^t N_t \subseteq N_n$ for all n, hence M is a left Quasi-Artinian.

(c)$\Rightarrow$(d) Let ς be a non-empty collection of left R-submodules of M such that for each $N \in \varsigma$ and $m \in \mathbf{Z}^+$, there exists $K \in \varsigma$ such that $K \subset N$, but $R^m N \not\subseteq K$. Now Let $N_1 \in \varsigma$ then there exists $N_2 \in \varsigma$ such that $RN_1 \not\subseteq N_2$,where $N_1 \supset N_2$, but $N_2 \in \varsigma$ hence there exists $N_3 \in \varsigma$, such that $R^2 N_2 \not\subseteq N_3$, where $N_1 \supset N_2 \supset N_3$ continuing in this manner we can construct an infinite descending chain $N_1 \supset N_2 \supset ... \supset N_n \supset ...$ of left R-submodules of M such that $R^m N_m \not\subseteq N_{m+1}$ m= 1,2,… , hence $R^m N_m \not\subseteq N_n$ for some n, which is a contradiction.

(d) $\Rightarrow$(a) Let ς be a non-empty collection of left R-submodules of M such that $RK \in \varsigma$ for all $K \in \varsigma$. Then $R^m K \in \varsigma$, for all $m \in \mathbf{Z}^+$. But $R^m K \subseteq K$ for all $m \in \mathbf{Z}^+$, hence by (d), there exists an $s \in \mathbf{Z}^+$ such that $R^s K \subseteq R^m K$ for all $m \in \mathbf{Z}^+$. Therefore if $m \geq s$, then $R^s K = R^m K$, and ς has a minimal element.

Now, we regard R as left R-module , then we have the following which is classify the class of left Quasi-Artinian ring .

Corollary 2.1.6

Let R be a ring then , the following conditions are equivalent :

(a) In each non-empty collection ς of left ideals of R such that if $J \in \varsigma$ then , $RJ \in \varsigma$ there exists a minimal element .

(b) For every descending chain of left ideals $I_1 \supseteq I_2 \supseteq \ldots$ there exists $m \in \mathbf{Z}^+$ such that , the descending chain $R^m I_1 \supseteq R^m I_2 \supseteq \ldots$ is terminate .

(c) R is left Quasi-Artinian .

(d) For every non-empty collection ς of left ideals of R , there exists $I \in \varsigma$ and $m \in \mathbf{Z}^+$ such that , $R^m I \subseteq J$, for any $J \in \varsigma$, $J \subseteq I$.

Proof :

Take , $M = {}_R R$ the left regular R-module in 2.1.5

□

Next , we prove the following .

Theorem 2.1.7

Let M be a left R-module . If RM is left Artinian , then M is left Quasi-Artinian .

Proof :

Let $N_1 \supseteq N_2 \supseteq \ldots$ be a descending chain of left R-submodules of M ,

then $RN_1 \supseteq RN_2 \supseteq \ldots$ is a descending chain of R-submodules of RM.

But RM is left Artinian , hence there exists $s \in \mathbf{Z}^+$ such that , $RN_s = RN_n$ for all $n \geq s$. Therefore , $R^s N_s \subseteq RN_s \subseteq N_n$, for all n . Hence M is left Quasi-Artinian .

□

Remark 2.1.8

(a) If R is a ring with identity , then $RM=M$ and M is a left Artinian and so it is left Quasi-Artinian .

(b) The converse of Theorem 2.1.7 needs not to be true as the following example shows :

Let which $M = \begin{bmatrix} \mathbf{Q} & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix}$ then , by example 2.1.2 (c) M is left Quasi-Artinian R-module . But $RM = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & 0 \end{bmatrix} = R$, is not left Artinian by example 2.1.2 (b) .Hence RM is not Artinian .

Now, let $\mathcal{X}$ be the class of all left Quasi-Artinian rings and $\mathcal{M}$ be the class of all left Quasi-Artinian modules .

Next , we prove the following .

Theorem2.1.9

$\mathcal{M}$ is S-closed

Proof :

If A is a left submodule of N and N is a left sub-module of M, then A is a left submodule of M hence , if M is a left Quasi-Artinian , and N is an R-submodule of M, then N is a left Quasi-Artinian .Hence $\mathcal{M}$ is S-closed

Theorem 2.1.10

(a) $\mathcal{M}$ is Q-closed .

(b) $\mathcal{X}$ is Q-closed .

Proof :

(a) Let M be a left Quasi-Artinian R-module and N is submodule of M. Suppose $\pi : M \rightarrow M/N = \overline{M}$ is the natural homorphism of of left Quasi-Artinian module onto $\overline{M}$. Then $\overline{N}_1 \supseteq \overline{N}_2 \supseteq \ldots$ is a descending chain of submodules of $\overline{M}$, and $N_1 \supseteq N_2 \supseteq \ldots$ is a descending chain of R-submodules of M, where $N_i = \pi^{-1}(\overline{N}_i)$, but M is left Quasi-Artinian , then there exists $m \in \mathbf{Z}^+$ such that $R^m N_m \subset N_n$, for all n, by the fact that , π is an onto mapping we have ,

$\pi(N_K) = \overline{N}_K$. Hence , $R^m \overline{N}_m \subset \overline{N}_n$ for all n , therefore $\overline{M}$ is left Quasi-Artinian .

(b) $\mathcal{X}$ is Q-closed can be proved by taking $M = {}_R R$ in **(a)**

A partial converse of theorem 2.1.10 is stated belw .

Theorem 2.1.11

$\mathcal{M}$ is E-closed .

Proof:

Suppose that N be an R-submodule of M and N, $M/N \in \mathcal{M}$.

Let $N_1 \supseteq N_2 \supseteq \ldots$ be a descending chain of left R-submodules of M .

Then , $N_1 \cap N \supseteq N_2 \cap N \supseteq \ldots$ is a descending chain of R-submodules of N . But N is left Quasi-Artinian , hence there exists $s \in \mathbf{Z}^+$ such that ,

$N^s\,(N_s \cap N) \subseteq N_n \cap N$, for all n . Now

$N_1 + N/N \supseteq N_2 + N/N \supseteq \ldots$ is a descending chain of submodules

of M/N and M/N is left Quasi-Artinian therefore , there exists $k \in \mathbf{Z}^+$

such that $R^k\,(N_k + N/N) \subseteq N_n + N/N$, for all n .That is ,

$R^k\,(N_k + N) \subseteq N_n + N$, for all n . Now let $m = \max\{s , k\}$. Then

$R^m\,(N_m \cap N) \subseteq N_n \cap N$ and $R^m\,(N_m + N) \subseteq N_n + N$, for all n .

Now , $R^m N_m = R^m\,[N_m \cap (N_m + N)]$

$\subseteq [N_m \cap (N_n + N)]$ and by modular law ,
$= N_n + (N_m \cap N)$, for all n

Therefore ,

$$R^m\,(R^m N_m) \subseteq R^m\,[\,N_n + (N_m \cap N)\,] = R^m N_n + R^m\,(N_m \cap N)$$

$$\subseteq N_n + (N_n \cap N) = N_n \text{ , for all } n$$

Hence $R^{2m}N_{2m} \subseteq R^{2m}N_m \subseteq N_n$, for all n . Therefore M is left Quasi-Artinian .

□

As an immediate consequence of Theorem 2.1.10 & 2.1.11 , we have the following ,

Corollary 2.1.12

A finite direct sum of a left Quasi-Artinian modules is left Quais-Artinian .

Remark 2.1.13

(a) If R/I is left Quasi - Artinian for any nonzero ideal I of R then , R need not to be left Quasi-Artinian as the following example shows :

Let $R = (\mathbf{Z}, +, .)$, then every nonzero ideal of R is of the form $I = nZ$ and $R/I \cong Z_n \in \mathcal{X}$, where $n \in \mathbf{Z}^+$, but $R \notin \mathcal{X}$.

(b) $\mathcal{X}$ need not be S-closed as the following example shows :

Let $S = (\mathbf{Z}, +, .)$ and $R = (\mathbf{Q}, +, .)$, then S is a subring of R and $R \in \mathcal{X}$, but $S \notin \mathcal{X}$.

□

Theorem 2.1.14

$\mathcal{X}$ is I-closed .

Proof:

Let R be a left Quasi-Artinian ring and I is a left ideal of R , and let $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots$ be any descending chain of left ideals of I, then $IJ_1 \supseteq IJ_2 \supseteq \ldots \supseteq IJ_n \supseteq \ldots$ is a descending chain of left ideals of R . But R is left Quasi-Artinian , so there exists $m \in \mathbf{Z}^+$ such that $R^m(IJ_m) \subseteq IJ_n \subseteq J_n$ for all n . But $I^m \subseteq R^m$ then ,

$I^m(IJ_m) = I^{m+1}J_m \subseteq IJ_n \subseteq J_n$, for all n . Therefore $I^{m+1}J_{m+1} \subseteq J_n$, for all n . Hence I is left Quasi-Artinian .

□

Remark 2.1.15

$\mathcal{X}$ is not E-closed , as the following example shows ,

Let $R = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{R} \end{bmatrix} = \left(\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a \in \mathbf{Q} \ , b , c \in \mathbf{R} \right\}, +, . \right)$

Then , $I = \begin{bmatrix} 0 & \mathbf{R} \\ 0 & 0 \end{bmatrix} = \left(\left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbf{R} \right\}, +, . \right)$ is an ideal of R

and $I^2 = 0$. Therefore I is nilpotent hence , I is left Quasi-Artinian . But $R/I \cong \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R} \end{bmatrix} \cong \mathbf{Q} \oplus \mathbf{R}$ and $\mathbf{Q}$, $\mathbf{R}$ are left Quasi-Artinian rings hence , $R/I \cong \mathbf{Q} \oplus \mathbf{R}$ is a left Artinian .Therefore , R/I is left Quasi-

Artinian , but R is not left Quasi-Artinian by Corollary 2.1.4 (d) .

However **, we have the following :**

Theorem 2.1.16

A finite direct sum of left Quasi-Artinian rings is a left Quasi-Artinian.

Proof:

By induction , it is enough to prove the result for $t = 2$. So , let $R = R_1 \oplus R_2$ where R_1, R_2 are left Quasi-Artinian . Now suppose that $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a descending chain of left ideals of R , then $R_1 I_1 \supseteq R_1 I_2 \supseteq \ldots \supseteq R_1 I_n \supseteq \ldots$ is a descending chain of left ideals of R_1 and $R_2 I_1 \supseteq R_2 I_2 \supseteq \ldots \supseteq R_2 I_n \supseteq \ldots$ is a descending chain of left ideals of R_2 , but R_1 and R_2 are left Quasi-Artinian rings , then there exists r , s such that $R_1^r (R_1 I_r) \subseteq R_1 I_n \subseteq I_n$ and $R_2^s (R_2 I_s) \subseteq R_2 I_n \subseteq I_n$.

Hence if $m = \max\{r,s\}$, then $R_1^m (R_1 I_m) \subseteq R_1 I_n \subseteq I_n$, for all n and $R_2^m (R_2 I_m) \subseteq R_2 I_n \subseteq I_n$, for all n ,

(***Note that*** : $R^m = (R_1 \oplus R_2)^m$, Since $R_1 \cap R_2 = 0$, it follows that $R_1 R_2 = 0$ and $R_2 R_1 = 0$. Therefore $(R_1 \oplus R_2)^m = R_1^m \oplus R_2^m$. Thuse

$R^{m+1} I_m = R_1^m (R_1 I_m) + R_2^m (R_2 I_m) \subseteq I_n$, for all n . And

$R^{m+1} I_{m+1} \subseteq R^{m+1} I_m \subseteq I_n$, for all n . Hence R is left Quasi-Artinian .

□

Remark 2.1.18

The converse of Theorem 2.1.16 also hold for if $R = R_1 \oplus R_2$ and R is left Quasi-Artinian , then $R_1 \cong R_1 \oplus \{0\} \lhd R$ and $R_2 \cong \{0\} \oplus R_2 \lhd R$, therefore by Theorem 2.1.13 R_1 and R_2 are left Quasi-Artinian .

□

Next we prove the following which gives a partial converse of Theorem 2.1.10 .

Corollary 2.1.19

If $I_1,\ldots,I_n$ are ideals of R such that $\bigcap_{i=1}^{n} I_i = 0$ and R/I_i is left Quasi-Artinian for all i , then R is left Quasi-Artinian .

Proof :

Since $R/\bigcap_{i=1}^{n} I_i \cong R/I_1 \oplus \ldots \oplus R/I_n$ and $\bigcap_{i=1}^{n} I_i = 0$, then $R \cong R/I_1 \oplus \ldots \oplus R/I_n$ but R/I_i is left Quasi-Artinian for all $i=1 , \ldots , n$ Hence by Theorem 2.1.16 we have R is left Quasi-Artinian .

□

2.2 .The ideals structure and some classification

We start with the following which generalize Brauer Theorem concerning Artinian rings and idempotent elements .

Theorem 2.2.1

Let I be a non-zero non-nilpotent left ideal in a left Quasi-Artinian ring , then I contains a non-zero idempotent element .

To prove this we need the following Lemma which by itself has some independent interest .

Lemma 2.2.2

Let R be left Quasi-Artinian ring . Then every non-nilpotent left ideal of R contains a minimal non-nilpotent left ideal .

Proof :

Let I be a non-nilpotent left ideal of R and suppose that I does not contains a minimal non-nilpotent left ideal of R . Assume that $I_1 = I$, then

$0 \neq I_1^2 \subseteq RI_1 \subseteq I_1$ and RI_1 is not nilpotent , since I_1 is not nilpotent , therefore there exists a non-nilpotent left ideal $I_2 \subsetneqq RI_1 \subseteq I_1$. Hence $0 \neq I_2^3 \subseteq R^2 I_2$ and $R^2 I_2$ is not nilpotent . In this way we can find a non-nilpotent left ideal $I_n \subsetneqq R^{n-1} I_{n-1} \subseteq I_{n-1}$. Hence

$I = I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ is a descending chain of left ideals of R which is a contradiction the fact that R is left Quasi-Artinian . Therefore I contains a minimal non-nilpotent left ideal of R .

Proof of theorem 2.2.1:

Let I be nonzero non-nilpotent left ideal of R . Since R is left Quasi-Artinian , then by Lemma 2. 2 . 2 , I contains a minimal non-nilpotent left ideal say , K. Since $K^2 \neq 0$, then there exists $0 \neq x \in K$ such that $Kx \neq 0$. However $Kx \subseteq K$ and Kx is a left ideal of R , hence by minimilty of K we have $Kx = K$. Therefore there exists $e \in K$ such that $ex = x$ and since $e^2 x = ex$ we get that $(e^2 - e)x = 0$. Now , let $K_o = \{a \in K \mid ax = 0\}$, therefore K_o is a left ideal of R and $K_o \subsetneq K$ since , $Kx \neq 0$, for some $x \in K$ .Therefore we must have $K_o = 0$ and $(e^2 - e) \in K_o$. Hence $e^2 = e$. Since $ex = x \neq 0$ we have that $e \neq 0$. But $K \subsetneq I$, hence $e \in I$.

□

***Now*, by proving theorem 2.2.1 we have actually obtained a criterion charactraization of a left ideal to be nilpotent for left Quasi-Artinian ring .**

Corollary 2.2.3

If R is a left Quasi-Artinian , then every non-zero nil left ideal of R is nilpotent

.

Proof :

Let N be a nonzero nil left ideal of R and suppose that N is not nilpotent . Then by Theorem 2.2.1 there exists a nonzero idempotent element e and $e \in N$. Therefore e is nilpotent which is a contradiction . Hence N must be nilpotent .

□

Theorem 2.2.4

Let R be left Quasi-Artinian ring , then

If I be a nil left ideal of R , then $l(I)$ *ess* R

Proof :

Since I is nil , therefore by Corollary 2.2.3 I is nilpotent . Let K be a nonzero left ideal in R , then there exists $n \geq 1$ such that $KI^{n-1} \neq 0$, $KI^n = 0$. Thuse $KI^{n-1} \subseteq K \cap l(I)$ and so , $K \cap l(I) \neq 0$.

□

Corollary 2.2.5

Let R be left Quasi-Artinian , then $l(J(R))$ *ess* R .

We now study semi-prime left Quasi-Artinian rings . The next Theorem shows that the condition that the left ideal to be minimal in Theorem 1.1.10 is not essential for the conclusion . In fact every left ideal is principel , with an idempotent generator .

Theorem 2.2.6

Let R be a semi-prime left Quasi-Artinian ring and I be a nonzero left ideal of R, then $I = Re$, for some nonzero idempotent e in R.

Proof :

Since I is not nilpotent , it follows from Theorem 2.2.1 ,that I contains a nonzero idempotent element say , e . Let $l(e) = \{x \in I \mid xe = 0\}$ then the set of left ideals $L = \{l(e) \mid 0 \neq e^2 = e \in I\}$ is not empty . Now , if $l(e) \in L$, then $Rl(e) \subseteq I$, since I is a left ideal of R , then $re \in I$, where $r \in R$, $e \in I$, therefore $0 \neq re^2 = re \in I$, but R left Quasi-Artinian , hence by Corollary 2.1.6 , L has a minimal element $l(e_0)$, say . Either $l(e_0) \neq 0$ or $l(e_0) = 0$. If $l(e_0) \neq 0$, then $l(e_0)$ must have an idempotent e_1, say . By definition of $l(e_0)$, $e_1 \in I$ and $e_1 e_0 = 0$. Consider $e_2 = e_0 + e_1 - e_0 e_1$, then $e_2 \in I$ and is itself a non-zero idempotent element . Moreover , $e_1 e_2 = e_1(e_0 + e_1 - e_0 e_1) = e_1 \neq 0$, hence $e_2 \neq 0$. Now if $x \in l(e_2)$, then $xe_2 = 0$ and $x(e_0 + e_1 - e_0 e_1) = 0$. Therefore $x(e_0 + e_1 - e_0 e_1)e_0 = 0$ and $xe_0 = 0$. Therefore $x \in l(e_0)$ and $l(e_2) \subset l(e_0)$, since $e_1 \in l(e_0)$ and $e_1 \notin l(e_2)$ we have that $l(e_2) \neq l(e_0)$, which contradicts the minimality of $l(e_0)$. Therefore $l(e_0) = 0$. But $(x - xe_0)e_0 = 0$ for all $x \in I$, hence $(x - xe_0) \in l(e_0) = 0$ and $x = xe_0$ for all $x \in I$, which implies that $I = Ie_0 \subseteq Re_0 \subseteq I$. Hence $I = Re_0$.

□

The next result which is a corollary to Theorem 2.2.6 is worthy of emphasis .

Corollary 2.2.7

Any semi-prime left Quasi-Artinian ring is a semi-simple left Artinian .

Proof :

By Theorem 2.2.6 every non-zero left ideal of R is generated by a non-zero idempotent say , e . But we know that e acts as right identity for the left ideal $I = Re$, and since R is itself an ideal , hence R has an identity element .Therefore R is left Artinian . Now , by Corollary 2.2.3 $J(R)$ is nilpotent , and since R is a semi-prime ring , implies that $J(R) = 0$. Hence R is a semi-simple.

□

Next we describe left Quasi-Artinian rings using the non-commutative version of Wedderburns' fundamental Theorem

Theorem 2.2.8

A ring R is left Quasi-Artinian if and only if R is a direct sum of

left Artinian ring with identity and a nilpotent ring .

To prove this we need the following result .

Lemma 2.2.9

Let R be left Quasi-Artinian ring and N be the nil radical of R, then R/N is a semi-simple left Artinian ring .

Proof :

Since N is nilpotent and R/N is left Quasi-Artinian , it follows that R/N is a semi-prime left Quasi-Artinian . Therefore , by Corollary 2.2.7 , R/N is a semi-simple Artinian ring .

Proof of theorem 2.2.8:

Suppose that R is a direct sum of a left Artinian ring with identity and a nilpotent ring , since any left Artinian ring and any nilpotent ring is left Quasi-Artinian , it follows by Theorem 2.1.16 that R is a left Quasi Artinian ring .

To prove the converse . Let $N = N(R)$ be the nil radical of R . Then by Corollary 2.2.3 , N is nilpotent and by Lemma 2.2.9 R/N is a semi-simple Artinian ring . Therefore by Wedderburns' Theorem R/N is a finite direct sum of its minimal ideals , each of which is a simple left Artinian ring , that is $R/N \cong \overline{N}_1 \oplus \overline{N}_2 \oplus \ldots \oplus \overline{N}_n$, where $\overline{N}_i = \langle \overline{e}_i \rangle$ is a minimal ideal of R/N which is a simple left Artinian ring . But a finite direct sum of left Artinian is again left Artinian , hence $\bigoplus_{i=1}^{n} \overline{N}_i$ is left Artinian ring . But $\overline{N}_i$ is a semi-simple left Artinian therefore , it has an identity element. Therefore $\bigoplus_{i=1}^{n} \overline{N}_i$ is left

Artinian ring with identity . Hence R is a direct sum of left Artinian ring with identity and nilpotent ring .

□

Theorem 2.2.10

If R is a semi-prime left quasi-Artinian and $I = Re = eR$ is an ideal of R , e an idempotent element, then any left (right, two-sided) ideal of I is also a left (right, two-sided) of R .

Proof :

Suppose that J is an arbitrary left ideal of I , considered as a ring . can Since $J \subseteq eR$, each element $a \in J$ be written in the form $a = er$ with $r \in R$; but then $a = er = e(er) = ea \in eJ$ leading to the equality $J = eJ$. Knowing this , $RJ = R(eJ) = (Re)J = IJ \subseteq J$ .Therefore J is a left ideal of R .

□

The last Theorem it is important its significance is explained in the following Corollary .

Corollary 2.2.11

Let R be a semi-prime left Quasi-Artinin ring . Then

(a) Each ideal of R is itself a semi-prime left Quasi-Artinian ring .

(b) Any minimal ideal of R is a simple ring .

Next **, we prove the following which characterizes the prime radical in a left Quasi-Artinian rings .**

Theorem 2.2.12

Let R be left Quasi-Artinian ring and I be a minimal ideal in R . Then $l(I)$ is a maximal ideal .

To prove this we need the following

Lemma 2.2.13

If R be left Quasi-Artinian ring , then every prime ideal of R is maximal .

Proof :

Let P be a prime ideal of R , then R/P is a prime ring . Now R/P is a semi-prime left Quasi-Artinian ring . Therefore by Corollary 2.2.7 R/P is a semi-simple left Artinian . Hence by Wedderburn's Theorem R/P is a finite direct sum of minimal ideals , each of which is a simple left Artinian ring . But a prime ring cannot be written as a direct sum of non-trivial ideals , hence R/P is a simple ring . Therefore P is maximal ideal .

□

Proof of theorem 2.2.12

By Lemma 2.2.13 , it is enough to show that $l(I)$ is a prime ideal in

R .Let $x, y \in R$ such that $x, y \notin l(I)$. Then $xI \neq 0$ and $yI \neq 0$, but $xI \subseteq I$ and $yI \subseteq I$ and since I is a minimal ideal of R, hence $xI = I$ and $yI = I$. Therefore $0 \neq xy \in I$ and $xyI \neq 0$. Hence $xy \notin l(I)$, and l(I) is a prime ideal of R .

□

Corollary 2.2.14

Let R left Quasi-Artinian ring . Then

$$J(R) = rad(R) = N(R) .$$

Theorem 2.2.15

If R be left Quasi-Artinian ring , then there exists only a finite number of distinct proper prime ideals of R .

Proof:

Suppose that , there exists an infinite sequence $\{ P_i \}$ of distinct proper prime ideals of R . Then $P_1 \supseteq P_1P_2 \supseteq \ldots \supseteq P_1 \ldots P_n \supseteq \ldots$ is a descending of ideals of R . Since R is left Quasi-Artinian , then there exists $m \in \mathbf{Z}^+$ such that the descending chain

$R^mP_1 \supseteq R^mP_1P_2 \supseteq \ldots \supseteq R^mP_1 \ldots P_n \supseteq \ldots$ is terminate . That is ,

$R^mP_1P_2 \ldots P_n = R^mP_1P_2 \ldots P_nP_{n+1}$, it follows from this that ,

$R^m P_1 \ldots P_n \subseteq P_{n+1}$, then $(P_1 \ldots P_n)^2 \subseteq R^m P_1 \ldots P_n \subseteq P_{n+1}$. But since P_{n+1} is prime then $P_1 \ldots P_n \subseteq P_{n+1}$ therefore $P_k \subseteq P_{n+1}$, for some $k \le n$ and by Lemma 2.2.13 P_k is maximal ideal of R so that we have , $P_k = P_{n+1}$. Contrary to the fact that the P_i are distinct ..

□

2.3 The submodules structure and some classiffication

we start with following :

Theorem 2.3.1

If $M = N+B$, where N and B are left Quasi-Artinian , then M is left Quasi-Artinian .

Proof :

Since $M = N+B$, we have $M/N = N+B/N \cong B/N \cap B$ which is homomorphic image of left Quasi-Artinian R-submodule. .Therefore M/N and N are left Quasi-Artinian . Hence by Theorem 2.1.11 M is left left Quasi-Artinian .

□

Theorem 2.3.2

If R left Quasi-Artinian ring , and M is a finitely generated left R-module , then M left Quasi-Artinian .

Proof

Let M be a finitely generated left R-module , then

$M = Rx_1 + Rx_2 + \ldots + Rx_n$, where $0 \neq x_i \in M$, $1 \leq i \leq n$ .If $n=1$

then , M is cyclic and therefore isomorphic to

${}_R R/L$ where $L = \{a \in R \mid ax_1 = 0\}$. Since ${}_R R$ is left Quasi-Artinian so, is every factor module . Assume inductively that the Theorem holds for modules which can be generated by n-1 or fewer elements . Then Rx_1 left Quasi-Artinian and

$$M/Rx_1 \cong (Rx_1 + Rx_2 + \ldots + Rx_n)/Rx_1$$

$$\cong (Rx_2 + \ldots + Rx_n)/Rx_1 \cap (Rx_2 + \ldots + Rx_n)$$

which is left Quasi-Artinian , by induction and Theorem 2.1.11 M is left Quasi-Artinian.

□

Theorem 2.3.3

Let R be a left Quasi-Artinian ring and M be a left R-module then ,

(a) *Soc(M) ess M*

(b) *Rad (M)* small in M

Proof

(a) Let $0 \neq x \in M$. Then , $\rho_x : R \rightarrow \rho_x(r) = rx \quad (r \in R)$ is a homomorphism of R onto the submodule Rx with *Kernal* $Ker\, \rho_x = l_R(x) = \{ r \in R \mid rx = 0 \}$. So $R/l_R(x) \cong Rx$ .But R is a left Quasi-Artinian,then by Theorem 2.1.10 Rx is left Quasi-Artinian

We claim that Rx contains a minimal submodule . To prove this let $\varsigma = \{ N \subseteq Rx \setminus 0 \neq x \in M , N \leq M \}$be a nonempty collection of R-submdules of Rx . and $J \in \varsigma$. Then , $J = Ry$ for some $0 \neq y \in M$ but $RJ = R(Ry) = (RR)y = R^2 y \subseteq Ry = J \in \varsigma$, and by Theorem 2.1.5 we have ς has a minimal element .Thus $Soc(Rx) \neq 0$

But $Soc(Rx) = Rx \cap Soc(M) \neq 0$, hence $Soc(M)\ ess\ M$.

(b) **First** ,we show that $Rad(M) = JM$, where $J = J(R)$ .Since for any left R- module M the factor module $Rad\left(M/Rad(M) \right) = 0$. Therefore , $M/RadM$ is subdirect product of simple left R- modules . But since , $J(R)$ is annihilates all simple left R-modules , so it annihilate $M/Rad(M)$ that is , $JM \leq Rad\,(M)$

Conversely , by Lemma 2.2.9 and Corollary 2.2.7 R/J is semi-simple then by Remark 1.1.34 we have ,

$$Soc(M) = r_M(J)$$

Therefore, $Soc(M/JM) = r_{M/JM}(J(R/J)) = r_{M/JM}(0) = M/JM$

Hence by Theorem 1.1.24 M/JM is semi-simple R/J - module

Since $J \subseteq \text{ann}$ (simple R - submodule of M) , then by Remark 1.1.17 we have M/JM is semi-simple R-module , thuse

$Rad(M/JM) = 0$ but $Rad(M/Rad(M)) = 0$. Therefore ,

$Rad(M) \leq JM$. Hence , $Rad(M) = JM$.

Now since , R left Quasi-Artinian assume $J^n = 0$ for some $n \in \mathbf{Z}^+$ and consider an R-submodule K of M with $JM + K = M$. Multiplying with J we obtain ,

$$J^2M + JK = JM \text{ , then } J^2M + JK + K = M$$

Continue in this way we have after n steps , $K = J^nM + K = M$.

Hence JM small in M therefore by first part , $Rad(M)$ small in M .

Corollary 2.3.4

Let R be left Quasi-Artinian ring and M left R-module , then M is finitely generated if and only if $M/Rad(M)$ is finitely generated .

Proof :

By Theorem 2.2.3 , since $Rad(M)$ small in M then the prove follows from Theorem 1.1.27 .

***Next*, we give another characterization of left quasi-Artinian ring , Namely the following :**

Theorem 2.3.5

N is ´ Let R be a ring , $N = N(R)$ be the nil radical of R then , R is a left Quasi-Artinian if and only if nilpotent and each of the

R/N , N/N^2 , N^2/N^3 , ... is left Quasi-Artinian R-modules . .

Proof :

Suppose R is left Quasi-Artinian then , by Corollary 2.2.3 N is nilpotent . Now , let $M = R$ be a left R- module then , M is left Quasi-Artinian R-module and N^i is an ideal of R for all i . Therefore , N^i is an R-submodule of M for all i , but by Theorem 2.1.10 R/N^i is left Quasi-Artinian for all $i \geq 1$. Also, N^i/N^{i+1} is R-submodule of R/N^{i+1} , So each N^i/N^{i+1} is left Quasi - Artinian .

To prove the converse , note that

Since $R/N \cong \dfrac{R/N^2}{N/N^2}$, it follows from Theorem 2.1.11 that R/N^2 is left Quasi-Artinian R-module and by induction R/N^i is left Quasi-Artinian for all i. But N is nilpotent, hence there exists $m \in \mathbf{Z}^+$ such that $N^m = 0$, therefore $R \cong R/N^m$ is left Quasi-Artinian R-module. Hence R is left Quasi-Artinian ring.

□

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شكـــر وتقـــدير

بسم الله الرحمن الرحيم . . . والصلاة والسلام على أفضل الأنبياء والمرسلين سيدنا محمد وعلى آله وصحبه أجمعين . .

أول الحمد وآخره ومبدأه ومنتهاه لربي الكريم سبحانه على ما منَّ به علي من نعم عظيمة ظاهرة وباطنة لا تعد ولا تحصى فلله الحمد حمداً كثيراً طيباً مباركاً فيه كما يحب ربنا ويرضى . وبعد ,,

فلما كانت أمتنا قد تأخرت عن دورها الريادي, ومكانها القيادي في مسيرة الركب الحضاري, وبعد أن كانت حادية ودليلة, ومقدمةٌ ونبيلة, تعثر خطوها وتثاقل سيرها حتى غدا حبواً . .

بعد أن كان الجميع لا يرون إلا غبار عدوها نحو المعالي, وارتفاع مكانها في العوالي . . . فإذا بزمرة من أبنائها البررة – عامل في مصنعة, وطبيب في عيادتة, وباحث في معهده, وتاجر في متجره, وصيدلي في مخبره, ومعلم في قاعته, ومفكر بين كتبه, وإعلامي على شاشته . . .

كل منهم يوقد في مجاله شمعة أمل في هذا الظلام الكئيب, والإحباط المطبق, ليزيل بعضاً من وحشة هذا الليل الطويل, ويبعث روحا من أنوار التفاؤل تسري وتبشر ببدر مضئ يتلوه الفجر الباسم الوضئ . .

ولما كان الإبداع هو ماء الحياة لأي نهضة, وومضة الإشغال لمحرك كل انطلاقة حضارية, وعنصر اللياقة في كل وثبة تاريخية . . فقد نقبت في صورة فوجدت منها الانعطاف إلى أصل مهجور أو فكرة قديمة أو معنى عام, وإعادة بعثه ببث التراب عنه أو بمزيد التأمل في أبعاده, ثم البناء عليه, والتفريع منه, وتخصيص عامه أو تعميم خاصة, والبحث في علاقته بغيره, وإيجاد الروابط ونحو ذلك مما يخدم الإفادة منه . .

فأملت أن يكون هذا البحث محاولة مني للإسهام في المشروع الحضاري الكبير ونقطة من نقاط هذا السيل القادم, حيث اتجهت في هذا البحث إلى بعض أنواع الحلقات وهي الحلقات شبة الأرتينية والحلقات المتلاشية, وقمت

بدراسة خصائصها وخصائص الحلقيات المعرفة عليها , وقد تمكنت بفضل الله وتوفيقة من تعميم أهم النظريات في الحلقات الأرتينية بالإضافة إلى تعميم بعض النتائج المترتبة عليها , وذلك بعد عدة محاولات لتجاوز عوائق التعميم ومناقشة العديد من الأفكار للوصول لهذا الهدف فوفقني الله لذلك من خلال إضافة بعض الشروط التي أتاحت لي الفرصة للتمكن من تجاوز هذه العوائق وتحقيق هذا التعميم , كما تمكنت من إيجاد العلاقة والرابط بين الحلقات الأرتينية والحلقيات شبة الأرتينية فلله الحمد من قبل ومن بعد (قل بفضل الله وبرحمته فبذلك فليفرحوا هو خير مما يجمعون)

وإن من تمام نعم الله عليَ – وهي لا تحصى – أن أكرمني بأن نلت شرف الرعاية الكريمة, والعناية الأبوية, من أحد رواد علم الرياضيات الحديثة وأفذاذ الأذكياء في أمتنا وطليعة نهضتنا , وأحد أهم مراجع هذا العلم وأعمدته, وقد أغدق عليَ من سعة علمه, ومديد تجربته, ودماثة خلقه, وثاقب رأيه, وبديع أفكاره ما فتح لي أفاقاً رحبة من المعرفة . وقد أوصى أسلافنا بالأخذ عن الأكابر في كل فن, فكانت منحة الله لي بأن أعلى سندي, وشد عضدي, بمباشرة الأخذ عن أستاذي الفاضل – تدريساً وإشرافاً – أ . د . فالح الدوسري حفظه الله وأمد في عمره ونفع بعلمه .

كما لا يفوتني في هذا المقام أن أزجي من الشكر أجزله ومن الثناء أفضله, لكل من لهم سهم في تقديم يد العون والتشجيع لي, وعلى رأسه – وعلى الرأس مكانه – المربي والمؤدب والمشجع المثالي, والحادب الحريص والدي الغالي, الذي رعى مسيرتي وأمدني بأسباب المواصلة عند كل تراجع في عزيمة أو ضعف في همة, فإذا به يربت عليَ, ويبث فيَ من بواعث الأمل وعزائم الهمة والعمل ما يطلق قيدي, ويكثر من الفوائد صيدي .

ثم لوالدتي الكريمة التي أمر الله بشكرها بعد شكره في قوله (أن اشكر لي ولوالديك) على حسن تربيتها وكريم رعايتها وصادق نصحها اسأل الله أن يحييها حياة طيبة وأن يكتب لها ولوالدي مثل أجر كل عمل صالح عملته واعمله انه جواد كريم .

كما أقدم شكري الخالص, وامتناني الكبير إلى أستاذي الأبلغ تأثيراً فيَ سعادة د . خالد فيلالي, حيث كان أول من علمني لغة هذا العلم وكيفية فك رموزه, وكان كثير الحث لنا على التعاطي مع المادة

الرياضية بلغتها الانجليزية مما أثر بعد ذلك عليَ إيجابياً في سهولة التعامل مع المراجع التي هي في مجملها باللغة الإنجليزية.

وأتقدم بالشكر والتقدير لأسرة كلية العلوم التطبيقية وعلى رأسها قائد مسرتها وربانها الموفق, صاحب الصدر الواسع, والأخلاق الدمثة, والتعامل الراقي المرن, سعادة عميد كلية العلوم التطبيقية أ. د. أحمد الخماش والذي كان صاحب المعروف الذي لا أنساه وأقدره وأثمنه بإتاحته الفرصة لي ودعمه القوي وموقفه النبيل من انضمامي لأعضاء هذا القسم الموقر, كما تتابع فضله وتوالى نبله ببابه المفتوح وإسدائه النصح وتقديم المعونة كلما احتجت لذلك في خلق كريم وتواضع جم فجزاه الله عني خير ما يجزي معلماً عن تلميذه ورئيساً عن مرؤوسه.

ولا أنسى تقديم الشكر والتقدير لكل من درس لي في هذه المرحلة وعلمني ما ساعدني على إتمام هذا البحث, وأخص بالشكر كلاً من سعادة د. سهل البار, د. صالح عبدالعزيز, و د. محمد زبير خان والذي تكبد بعد ذهابه للهند فأرسل لنا مرجع لم نجده هنا, و أ. سميرة طيب, ولأعضاء قسمي الموقر, ابتداءاً من رؤساء القسم السابقين د. محمد الدويخ, د. عبدالفتاح قارئ, ووكيلتهم د. ابتسام أبو سليمان وزملائي وزميلاتي أعضاء هيئة التدريس بالقسم. كما أخص بالشكر سكرتارية القسم متمثلة في الأختين الكريمتين / حياة بلجون وعائشة الحازمي. كما لا أنسى أن اشكر القائمين على مدينة الملك عبدالعزيز للعلوم والتقنية لما قدموه من تسهيلات في توفير بعض البحوث من خارج المملكة. كما أشكر كلاً من سعادة أ.د. كارل فريدريك بيرغ من جامعة **Norwegian** وسعادة أ.د. جون بيشي من جامعة نورثن الينوس بأمريكا على جميل تعاونهما بإرسال بعض التوضيح لبعض استفسارات لدي وإرسالهم بحوث لي على رغم بعد المسافة بيننا.

ولا أستطيع هنا إلا أن اعبر عن عظيم امتناني وتقديري لصديقتيَ أماني الفضلي وإيمان اللقماني اللتان تقاسمتا معي أيام هذه المرحلة بساعتها ودقائقها وما فيها من إرهاق وجهد وجد وكفاح وآمال فلم تبخلا عليَ من دعمهما وودهما ولطفهما حتى تيقنت بالفعل أنهما أثمن ما حصلت عليه في هذه المرحلة فليحفظهم الله وليديم عليهما نعمه.

وأخيراً ليس بآخر فأنا اشكر كل من أعانني فأوضح لي غامض أو صحح لي خطأ أو آعارني كتاباً أو دلني على فائدة فمعروفهم عندي وإن قصرت عن مكافأتهم محفوظٌ غير مضيع ومشكورٌ غير مكفور فجزاهم الله عني خير الجزاء وأوفاه .

اللهم رب جبرائيل وميكائيل وإسرافيل فاطر السموات والأرض عالم الغيب والشهادة أنت تحكم بين عبادك فيما كانوا فيه يختلفون اهدني لما اختلفت فيه من الحق بإذنك إنك تهدي من تشاء إلى صراط مستقيم .

سبحان ربك رب العزة عما يصفون والحمد لله رب العالمين وصلى الله على سيدنا محمد وعلى آله وصحبه أجمعين .

بسم الله الرحمن الرحيم

خلاصة البحث

يعتبر تحديد بنية الحلقيات (الحلقات) الأرتينية واحدة من أهم المشاكل في نظرية الحلقيات (الحلقات) . وقد دلت الدراسات المتتالية لتلك الحلقيات و الحلقات على إمكانية دراسة حلقيات و حلقات أعم . وتناولنا في هذا البحث دراسة نوع جديد من الحلقيات و الحلقات يسمى الحلقيات و الحلقات شبه الأرتينية اليسرى والذي يملك العديد من الخواص التي تعتبر تعميم للخواص الأساسية للحلقيات (الحلقات) الأرتينية اليسرى والحلقات المتلاشية.

والشرط الذي تحققة هذه الحلقيات (الحلقات) هو:

إذا كانت R حلقة, M حلقية يسرى على الحلقة R فيقال عن M أنها حلقية شبه أرتينية يسرى (***left Quasi-Artinian***) إذا كان لأي سلسلة متنازلة من الحلقيات الجزئية اليسرى من M يوجد عدد صحيح موجب m بحيث أن : $R^m N_m \subseteq N_n \quad \forall n$. وحيث أن R حلقية يسرى على نفسها فإن R تسمى حلقة شبه أرتينية يسرى. وهذا النوع من الحلقات هو تعميم للحلقات الأرتينية اليسرى والحلقات المتلاشية.

وتضم هذه الرسالة فصلين احتوى **الفصل الأول** منها على مجموعة من التعاريف والخواص الأساسية بالإضافة إلى بعض النتائج المعروفة في نظرية الحلقيات (الحلقات) والتي احتجنا إليها خلال البحث.

أما **الفصل الثاني** في هذه الرسالة **فقد ضم ثلاثة بنود** تناولنا في **البند الأول** منها مفهوم الحلقيات (الحلقات) شبه الأرتينية اليسرى بالإضافة إلى عدد من الأمثلة على هذا النوع من الحلقيات (الحلقات) . ومن ثم أوجدنا الشروط المكافئة لتعريف الحلقيات (الحلقات) شبه الأرتينية اليسرى. ثم درسنا العلاقة بين الحلقيات الأرتينية اليسرى والحلقيات الشبه الأرتينية اليسرى وبصورة خاصة أثبتنا في مبرهنة (2.1.7) أنه إذا كانت RM حلقية أرتينية يسرى فإن M حلقية شبه أرتينية يسرى .

وبعد ذلك بيَّنا في مبرهنة (2.1.9) أن الحلقيات شبه الأرتينية اليسرى مغلقة بالنسبة للأجزاء (S-closed) أي أنه إذا كانت M حلقية شبه أرتينية يسرى على الحلقة R و N حلقية جزئية منها فإن N حلقية شبه أرتينية يسرى. كما أنها مغلقة بالنسبة لعملية القسمة (الباقي) (Q-closed) أي أنه إذا كانت M حلقية شبه أرتينية يسرى و N حلقية جزئية منها فإن M/N حلقية شبه أرتينية يسرى. بالإضافة إلى أننا أثبتنا في مبرهنة (2.1.11) أن الحلقيات شبه الأرتينية اليسرى مغلقة بالنسبة للتوسع (E-closed) أي أنه إذا كانت M حلقية يسرى على الحلقة R و N حلقية جزئية منها بحيث أن كلاً من N و M/N حلقية شبه أرتينية يسرى فإن M حلقية شبه أرتينية يسرى . وبعد ذلك أثبتنا أن الحلقات شبه

الأرتينية اليسرى ليست مغلقة بالنسبة للأجزاء والتوسع ولكنها مغلقة بالنسبة للمثاليات اليسرى (I-closed) أي أنه إذا كانت R حلقة شبه أرتينية يسرى و I مثالية يسرى من R فإن I شبه أرتينية يسرى. كما أنها مغلقة بالنسبة لعملية القسمة (الباقي) . وأخيرا أثبتنا في مبرهنة (2.1.16) أن الجمع المباشر لمجموعة منتهية من الحلقات شبه الأرتينية اليسرى يكون حلقة شبه أرتينية يسرى.

أما في البند الثاني في هذه الرسالة فقد خصص لدراسة بنية المثاليات في الحلقات شبه الأرتينية اليسرى. فقد أثبتنا في مبرهنة (2.2.1) أن كل مثالية غير متلاشية في حلقة شبه أرتينية يسرى تحوي عنصر متعادل. ثم أثبتنا في مبرهنة (2.2.6) أنه إذا كانت R حلقة شبه أولية وشبه أرتينية يسرى فإن كل مثالية غير صفرية تكون مولدة بعنصر متعادل . ثم أعطينا تصنيف للحلقات شبه أرتينية اليسرى في الحالة الأبدالية حيث أثبتنا في مبرهنة (2.2.8) أنه إذا كانت R حلقة إبدالية فإن R حلقة شبه أرتينية يسرى إذا وإذا فقط كانت R جمع مباشر لحلقة أرتينية يسرى ذات عنصر محايد وحلقة متلاشية . وأثبتنا كذلك في مبرهنة (2.2.12) أنه إذا كانت R حلقة إبدالية شبه أرتينية يسرى و I مثالية صغرى من R فإن تالف المثالية الأيسر($l(I)$) يكون مثالية عظمى في R . ثم بينا أنه إذا كانت R حلقة أرتينية يسرى فإنه يوجد عدد منتهي من المثاليات الأولية الفعلية المختلفة.

أما البند الثالث في هذا الفصل فخصص لدراسة بنية الحلقيات الجزئية من الحلقيات المعرفة على حلقات شبه أرتينية يسرى حيث أثبتنا في مبرهنة (2.3.2) أنه إذا كانت R حلقة شبه أرتينية يسرى و M حلقية يسرى على R فإن كل حلقية يسرى M على R ذات مولدات منتهية تكون حلقية شبه أرتينية يسرى . ثم أثبتنا في مبرهنة (2.3.3) أن $Soc(M)$ لهذا النوع من الحلقيات يكون حلقية كبرى في M و أما جذر الحلقية M فيكون حلقية جزئية صغيرة في M . وأخيراً أعطينا تصنيفاً أخر للحلقيات (الحلقات) شبه الأرتينية اليسرى وبصورة خاصة أثبتنا في مبرهنة (2.3.5) أن الحلقة R تكون شبه أرتينية يسرى إذا وإذا فقط كان الجذر شبه متلاشي للحلقة R , $N=N(R)$ مثالية متلاشية وكل من

R/N , N/N^2 , N^2/N^3 , ... حلقيات شبه أرتينية يسرى على الحلقة R .

وأخيراً, نود أن نشير إلى أنه تم إرسال بحثين للنشر من هذه الرسالة,

والله الموفق لمايحبه ويرضاه

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الحلقات والحلقيات شبة الأرتينية اليسرى

الرسالة المقدمة من

أميمة بنت مصطفى بن محمد الأمين الشنقيطي

كجزء متمم للحصول على درجة
الماجستير في الرياضيات
(الجبر)

أشراف

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